AUTOMATIC DIAGRAMS IN GEOMETRY.

One of the great obstacles met with in the study of elementary geometry as presented in the current text-books is the needle-in-the-haystack method of presentation of the symbols used in the diagrams. For instance, every angle under discussion requires for its recognition in the diagram three acts of perception (recognition of three letters). The designation of an angle in the argument requires the recognition of three symbols in the text and the selection of three more from among the many presented in the diagram. If the concept could be symbolized by a single letter instead of by three, the labor of perception with its accompanying recognitions and judgments would be greatly reduced.

Under the current system, the student is bewildered by the wilderness of letters and his energies are taxed, not in grasping the concept or argument, but in trying to disentangle the mass of letters and in finding what concept they are intended to symbolize. He is so confused and worried by this needle-in-the-haystack puzzle that the relations between the concepts are not readily grasped. The mass of letters on the page is not readily visualized into a concept. How much clearer the concept and quicker the grasp and firmer the grip if instead of $< ACB$ the reader sees one letter, C.

I present here a system of notation which seeks to avoid the difficulties specified above. It has been tested in class-room use with great success, with scholars of all ages. It has been a common experience after presentation in this notation of a theorem ordinarily confused and complicated to hear the exclamation—Is that all it is! Why, that's easy!" The illustrations given below will show the foundation for this exclamation.

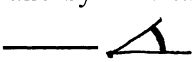
One of the gains is that from the diagram alone, in many cases, the student reads the proposition, the demonstration, and the conclusion; and in most cases with only a few lines of added

text. It saves the time of the student, as he has far less typography to interpret, far less lettering to do, and a minimum of explanatory context to make, either verbal or written.

It saves the time of the teacher, for a few seconds' inspection of the diagram tells as much of the student's idea of his hypothesis, argument, and conclusion as the same number of minutes devoted to the current method of demonstration. A glance at the diagram enables the teacher to put his finger on the exact break in the student's procedure without any question as to the student's mental processes; the diagram shows exactly the order of his different steps. A mistake in his premises is shown instantly. If his first step in demonstration or construction is wrong, however complicated the diagram may be, its detection is instantaneous.

Instead of a mass of letters without order or system — or worse, an arbitrary and haphazard order burdened on the student's memory in order to save the teacher's time — the number of letters is reduced to a minimum, the order is natural (alphabetic) and therefore no burden on the memory, and each letter opens up to the teacher the story of the student's mental procedure. There is no question of how this or that was done, or what was done next; the diagram, with an occasional addendum, speaks for itself. The energy of the student and teacher is concentrated on the logic of the subject and not wasted on the typographical interpretation. If clear type, good paper, etc., are a help, so also is a clear, simple and automatic notation.

Lines are designated by lower-case letters; *angles* by lower-case Greek letters (or where preferred, by English capital letters); *points* and *planes* by capital letters; *areas* by capital letters enclosed in a circle or rectangle or underscored.

Given parts are designated by heavy lines and by the *middle* letters of the alphabet beginning with *l*, *L*, λ, 

Construction parts are designated by light (or dotted) lines and lettered with the earlier letters of the alphabet, beginning with *a*, α, A according as the part is line, angle or otherwise.

Parts *given for discussion* or concerning which some statement

is made or required in a problem are designated by the latter letters of the alphabet, u, v, X, Y, ϕ, etc. An open dot ○ (small circle), would indicate a point under discussion.

Required parts in construction problems are indicated by being drawn in double lines.

Letters with *primes*, *seconds*, etc., indicate that the magnitudes they represent are the same. The primes, etc., are used to indicate difference of position only.

Letters with *subscripts* indicate different parts of the same magnitude, line, angle, etc., or parts so intimately connected that they may be considered as belonging to one operation; or parts whose order of construction is indifferent; or parts which follow immediately and closely in order of construction, as consequences of each other, the order, if any, being indicated by the subscripts.

Parallelism of lines is indicated by connecting the parallel lines with a dotted sigmoid line, arrowheaded at the ends and crossed by the sign of parallelism, .

Right angles are indicated by a small quadrant or a small isosceles right triangle at the vertex.

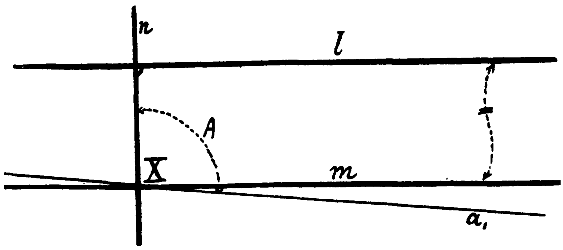
Where the length of a line is given but its position not specified, it is indicated by a light line with heavy arrow heads, $\longleftrightarrow$

Coincidence of lines, or congruency generally is indicated by $\equiv$.

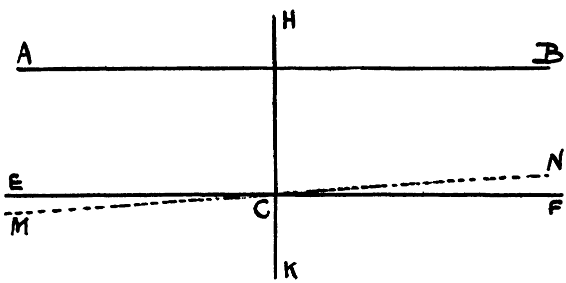
To illustrate the contrast of methods, the two systems of notation are presented in parallel columns, the automatic system, to show its potency, being condensed more than would perhaps be advisably in a text-book.

For typographical reasons English capital letters have been used instead of Greek letters for angles.

1) *A straight line perpendicular to one of two parallels is perpendicular to the other.*

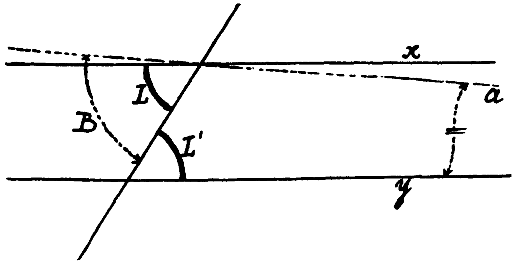


$A =$ rt. $\angle$
$\therefore a_1 \parallel l$
(two straight lines $\perp$ same line, etc).
$\therefore a_1 \equiv m$
(through one point only one $\parallel$, etc.)
$\therefore X =$ rt. $\angle$ Q. E. I.

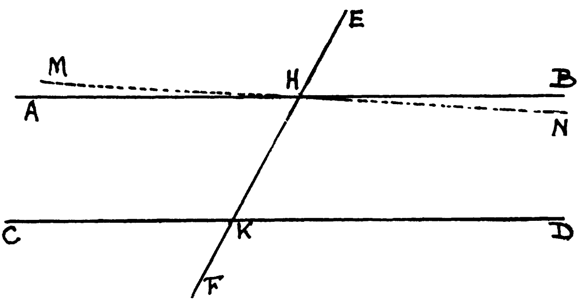


GIVEN lines AB and EF $\parallel$, and line HK $\perp$ to AB, and cutting EF at C.
TO PROVE $HK \perp$ to EF
PROOF. Suppose *MN drawn through C* $\perp$ to HK
Then MN is $\parallel$ to AB
(two straight lines $\perp$ same line, etc.)
But EF is $\parallel$ to AB
$\therefore EF$ coincides with MN
(through one point only one $\parallel$, etc.)
$\therefore$ EF is $\perp$ to HK
that is HK is $\perp$ to EF Q. E. D.

2) *Two straight lines cut by a transversal making the alternate-interior angles equal are parallel.*



$B = L'$ (alt. int. $\angle$ *s* of $\|$ lines)
$= L$ (by hyp.)
$\therefore a \equiv x$
But $a \|$ to y
$\therefore$ $x \|$ to y. Q. E. I.



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle AHK = \angle HKD$.

TO PROVE $AB \|$ to CD

PROOF. Suppose MN drawn through $H \|$ to CD.

Then $\angle MHK = \angle HKD$
(alt. int. $\angle$ *S* of $\|$ lines)

But $\angle AHK = \angle HKD$ Hno.

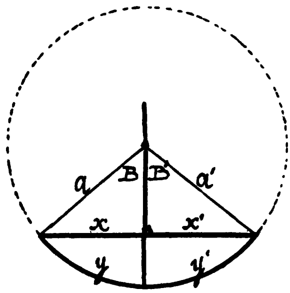
$\therefore$ $\angle MHK = \angle AH$K

$\therefore$ these lines MN and AB coincide.

But MN is $\|$ to CD

$\therefore$ AB which coincides with MN, is $\|$ to CD. Q. E. D.

3) *A radius perpendicular to a chord bisects the chord and the subtended arc.*



Since $a = a'$ $\therefore$ $x = x'$. Q. E. D.

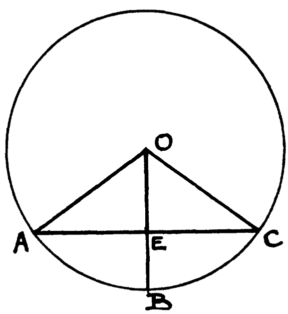
(= oblique lines cut off = segments, etc.)

$\therefore$ $B = B'$

(oblique lines cutting off = segments, etc.)

$\therefore$ $y = y'$. Q. E D.

(= angles subtend = arcs.)



GIVEN the radius OB perpendicular to the chord AC at E.

TO PROVE that $AE = EC$, $AB = BC$.

PROOF. Drawing radii OA, OC, then

$OA = OC$

and $AE = AC$. Q. E. D.

(= oblique lines cutting off = segments, etc.)

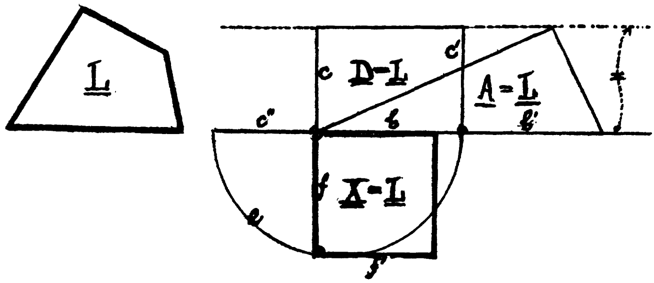
$\therefore$ $\angle AOE = \angle EOC$

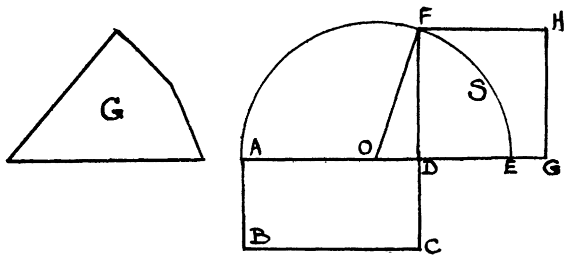
(oblique lines cutting off = segments, etc.)

$\therefore$ $AB = BC$ Q. E. D.

(= angles subtend = arcs.)

4) *To construct a square equal to a given polygon.*





GIVEN polygon G

REQUIRED to construct a square equal to G.

CONSTRUCTION. 1) Construct a triangle equal to G.

2) By drawing a line through the vertex of the triangle parallel to the base and erecting perpendiculars from an extremity and the middle point of the base, construct a rectangle, as $ABCD$ equal to this triangle

3) Produce AD to E making $DE = CD$.

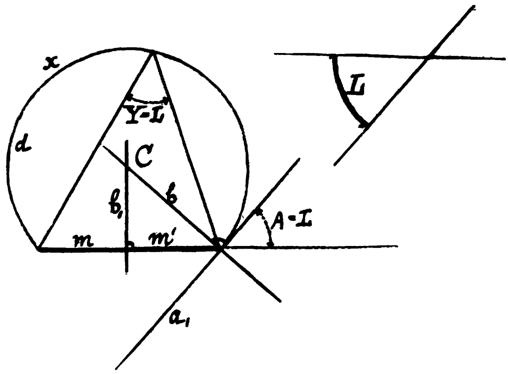
Bisect AE at O,

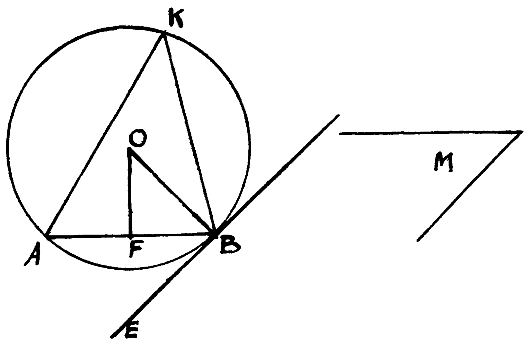
and with the center O, and radius OE, describe a semicircle.

4) Produce CD to meet the circumference at F and construct a square on DF.

Then DF^2, S in the figure, is the required polygon. Q. E. I.

5) *Upon a given straight line, to describe a segment of a circle in which a given angle may be inscribed.*





Let AB be the given line, and M the given angle.

Construct the $\angle\ ABE$ equal to the $\angle\ M$.

Bisect the line AB by the $\perp$ OF

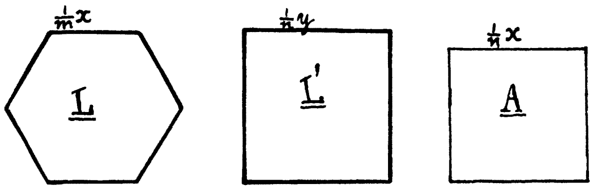
From the point B draw $BO \perp$ to EB.

From O the point of intersection of EO and BO, as a center, with a radius equal to OB, describe a circumference.

The segment AKB is the segment required.

A striking case is that bugbear of student and teacher alike, the proposition as to the perimeters of regular polygons of the same area. The teacher or student who has not become tangled up in that puzzling question as to whether it is the perimeters or the areas of Q and Q' and Q and Q'', etc., that are equal, is to be congratulated.

6) *Of regular polygons having a given area that which has the greatest number of sides has the least perimeter.*



(reg. m-gon) (reg. n-gon) (reg. n-gon)

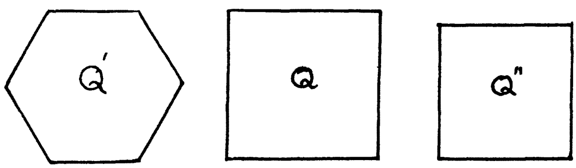
$$m > n.$$

ANALYSIS $L > A$

(of isoperimetric reg. polyg. the greater-sided has greater area)

$\therefore$ $L' > A$

$\therefore$ $y > x$ Q. E. I.



Let Q and Q' be regular polygons having the same area and let Q' have the greater number of sides.

TO PROVE the perimeter of $Q >$ the perimeter of Q'.

PROOF. Let Q'' be a regular polygon having the same perimeter as Q' and the same number of sides as Q.

Then $Q' > Q''$

(of isoperimetric reg. polyg. the greater-sided has greater area)

But $Q = Q'$

$\therefore$ $Q > Q''$

$\therefore$ perimeter of $Q >$ the perimeter of Q

But the perimeter of $Q' =$ the perimeter of Q''. Hyp.

$\therefore$ the perimeter of $Q >$ the perimeter of Q'. Q. E. D.

To ensure an understanding of the automatic diagrams the following explanatory annotation is added.

1) The heavy lines and letters *l*, *m*, *n* show that two parallel lines are given and a perpendicular to one of them (shown by the small quadrant). *X* shows what is under discussion. As the line A_1 has not yet been drawn there can be no ambiguity about *X*.

2) The heavy arcs and the letters *L*, *L'* show that the alternate interior angles are equal. The *x* and *y* show the elements

under discussion. Had they been marked x, x' it would have been an assertion of their equality, to be investigated.

3) The heavy lines and the small quadrant assert that we have an arc of a circle, a subtended chord and a radius perpendicular to it; the lettering $x = x'$, $y = y'$ asserts that the arc and chord are bisected, a proposition presented for investigation.

4) The heavy lines and the underscored L indicate a given area. The final $X = L$ shows that an equivalent square is required (sides, $f = f'$). The $b = b'$ shows the bisection of the base of A. $c = c' = c''$ tells its own tale.

5) The heavy lines and the letters m, m' indicate a given line which is to subtend an angle V in a required arc x, the required angle Y to be equal to a given angle L. The fact that L is given in size but in no specified position is shown by the heavy arc and the light sides of L. a_1 shows that this line is a close accompaniment of the first operation A, the laying off of an angle equal to L on the given line. b, b_1 show that the next operation is indifferently the drawing of the perpendicular b, or the erection of the bisecting perpendicular b_1.

6) That the two given areas are equal is shown by L, L', and that the lengths of the primeters is under investigation is shown by the letters x, y. A underscored shows that the first operation is to construct a polygon of the same number of sides as L', shown by the diagram and by the fact that the side of each is $\frac{1}{n}$ of the perimeters, x and y.

This system of notation sharply separates in the diagram those parts belonging respectively to the Hypothesis, the Construction, and the Conclusion. By reserving the heavy lines for those parts of the Hypothesis which do not belong to the Universe of Discourse, the differentiation can be made still more sharp, viz.: (1°) The Universe of Discourse, light lines and middle letters; (2°) the remainder of the Hypothesis, heavy lines and middle letters; (3°) the Construction, light lines and first letters; (4°) the Conclusion, light lines and latter-letters; or (5°) Required Construction, double lines and latter-letters.

Accompanying this system are a number of unimportant but simple and useful details, which add greatly to the legibility of the diagrams, but which are not really essential to the main features. These may be presented in a future article.

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